AA214 – Fall 1996 Solution Set 2

1. The second-order backward (or "upwind") differencing approximation to a 1st derivative given as a point operator

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{2\Delta x}\left(u_{j-2} - 4u_{j-1} + 3u_j\right)$$

is written in banded matrix form as

$$\frac{1}{2\Delta x}B(M:1,-4,3,0,0)$$

which is a lower triangular three banded matrix. It can be rewritten as the sum of the symmetric and skew symmetric matrices

$$\frac{1}{2\Delta x}((B+B^T)/2 + (B-B^T)/2)$$

where $B^T = B(M:0,0,3,-4,1)$ so we have

$$\frac{1}{4\Delta x}(B(M:1,-4,6,-4,1) + B(M:1,-4,0,4,-1))$$

Notice that we now have two terms which are scaled with $\frac{1}{4\Delta x}$.

2. The Taylor table for the symmetric matrix $B(M:1,-4,6,-4,1) \approx (?)_j$ is (for now leave off the $\frac{1}{4\Delta x}$)

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \cdot \left(\frac{\partial^5 u}{\partial x^5}\right)_j$	$\Delta x^6 \cdot \left(\frac{\partial^6 u}{\partial x^6}\right)_j$
u_{j-2}	1	$(-2)\cdot\frac{1}{1}$	$(-2)^2\cdot\frac{1}{2}$	$(-2)^3\cdot\frac{1}{6}$	$(-2)^4\cdot\frac{1}{24}$	$(-2)^5\cdot\frac{1}{120}$	$(-2)^6\cdot\frac{1}{720}$
$-4\cdot u_{j-1}$	-4	$-4(-1)\cdot\frac{1}{1}$	$-4(-1)^2\cdot\frac{1}{2}$	$-4(-1)^3\cdot\frac{1}{6}$	$-4(-1)^4\cdot\frac{1}{24}$	$-4(-1)^5\cdot\frac{1}{120}$	$-4(-1)^6\cdot\frac{1}{720}$
$6\cdot u_j$	6						
$-4\cdot u_{j+1}$	-4	$-4\cdot\frac{1}{1}$	$-4\cdot\frac{1}{2}$	$-4\cdot\frac{1}{6}$	$-4\cdot\frac{1}{24}$	$-4\cdot\frac{1}{120}$	$-4\cdot\frac{1}{720}$
u_{j+2}	1	$(2)\cdot\frac{1}{1}$	$(2)^2\cdot\frac{1}{2}$	$(2)^3\cdot\frac{1}{6}$	$(2)^4\cdot\frac{1}{24}$	$(2)^5\cdot\frac{1}{120}$	$(2)^6\cdot\frac{1}{720}$
$((?))_j$	0	0	0	0	1	0	$\frac{1}{6}$

Now we still have the $4\Delta x$ to divide out giving the derivative being approximated

$$(?) = \frac{\Delta x^3}{4}\cdot\left(\frac{\partial^4 u}{\partial x^4}\right)_j$$

which is a derivative scaled by Δx^3. More on this later.

The truncation error for this term is (remember to divide by the $4\Delta x$)

$$er_t = \frac{\Delta x^5}{24}\left(\frac{\partial^6 u}{\partial x^6}\right)_j$$

The Taylor table for the skew symmetric matrix $B(M: 1, -4, 0, 4, -1) \approx (?)_j$ is (again leave off the $\frac{1}{4\Delta x}$ for now)

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \cdot \left(\frac{\partial^5 u}{\partial x^5}\right)_j$
u_{j-2}	1	$(-2) \cdot \frac{1}{1}$	$(-2)^2 \cdot \frac{1}{2}$	$(-2)^3 \cdot \frac{1}{6}$	$(-2)^4 \cdot \frac{1}{24}$	$(-2)^5 \cdot \frac{1}{120}$
$-4 \cdot u_{j-1}$	-4	$-4(-1) \cdot \frac{1}{1}$	$-4(-1)^2 \cdot \frac{1}{2}$	$-4(-1)^3 \cdot \frac{1}{6}$	$-4(-1)^4 \cdot \frac{1}{24}$	$-4(-1)^5 \cdot \frac{1}{120}$
$0 \cdot u_j$	0					
$4 \cdot u_{j+1}$	4	$4 \cdot \frac{1}{1}$	$4 \cdot \frac{1}{2}$	$4 \cdot \frac{1}{6}$	$4 \cdot \frac{1}{24}$	$4 \cdot \frac{1}{120}$
u_{j+2}	-1	$-(2) \cdot \frac{1}{1}$	$-(2)^2 \cdot \frac{1}{2}$	$-(2)^3 \cdot \frac{1}{6}$	$-(2)^4 \cdot \frac{1}{24}$	$-(2)^5 \cdot \frac{1}{120}$
$((?))_j$	0	4	0	$-\frac{4}{3}$	0	$-\frac{7}{15}$

which gives after dividing out the $4\Delta x$, the derivative being approximated

$$(?) = \left(\frac{\partial u}{\partial x}\right)_j$$

which is a consistent approximation to the first derivative. and the

$$er_t = \frac{-\Delta x^2}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j$$

showing that this is a second order approximation to the first derivative. The symmetric term is also part of the error, but it is $O(\Delta x^3)$. It does contribute though to the understanding of how one sided difference approximations to the first derivative produce accurate (to some order) representations (the skew-symmetric part) with some form of dissipation (the symmetric part). The form of the added dissipation is detailed by the symmetric part and is consistent since as $\Delta x \to 0$ the error term goes to zero.

3. The 7x7 matrix operator that expresses the skew symmetric form with periodic (bc) can be written as

$$\begin{bmatrix} 0 & 4 & -1 & 0 & 0 & 1 & -4 \\ -4 & 0 & 4 & -1 & 0 & 0 & 1 \\ 1 & -4 & 0 & 4 & -1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 4 & -1 \\ -1 & 0 & 0 & 1 & -4 & 0 & 4 \\ 4 & -1 & 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

4. From the Taylor table

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$a \cdot \Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j-1}$		a	$a \cdot (-1) \cdot \frac{1}{1}$	$a \cdot (-1)^2 \cdot \frac{1}{2}$	$a \cdot (-1)^3 \cdot \frac{1}{6}$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$		1			
$-d \cdot u_{j-1}$	$-d$	$-d \cdot (-1) \cdot \frac{1}{1}$	$-d \cdot (-1)^2 \cdot \frac{1}{2}$	$-d \cdot (-1)^3 \cdot \frac{1}{6}$	$-d \cdot (-1)^4 \cdot \frac{1}{24}$
$-c \cdot u_j$	$-c$				
$-b \cdot u_{j+1}$	$-b$	$-b \cdot (1) \cdot \frac{1}{1}$	$-b \cdot (1)^2 \cdot \frac{1}{2}$	$-b \cdot (1)^3 \cdot \frac{1}{6}$	$-b \cdot (1)^4 \cdot \frac{1}{24}$

the following equation has been constructed to maximize the order of accuracy

$$\begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ -2 & -1 & 0 & -1 \\ 3 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

This has the solution $[a, b, c, d] = [2, 1, 4, -5]/4$, so the scheme can be expressed as

$$2(\delta_x u)_{j-1} + 4(\delta_x u)_j - \frac{1}{\Delta x}[-5u_{j-1} + 4u_j + u_{j+1}] = O(\Delta x^3)$$

The Taylor series error of this difference scheme is

$$\begin{aligned} er_t &= \left(2 \cdot (-1)^3 \cdot \frac{1}{6} + 5 \cdot (-1)^4 \cdot \frac{1}{24} - 1 \cdot (1)^4 \cdot \frac{1}{24}\right) \frac{\Delta x^3}{4} \left(\frac{\partial^4 u}{\partial x^4}\right)_j \\ &= -\frac{\Delta x^3}{24} \left(\frac{\partial^4 u}{\partial x^4}\right)_j \end{aligned}$$

5. For a 4 (interior) point mesh the above scheme in matrix form becomes

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} (\delta_x u)_1 \\ (\delta_x u)_2 \\ (\delta_x u)_3 \\ (\delta_x u)_4 \end{bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} 4 & 1 & 0 & 0 \\ -5 & 4 & 1 & 0 \\ 0 & -5 & 4 & 1 \\ 0 & 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \frac{1}{\Delta x} \begin{bmatrix} -5u_0 - 2\Delta x(\partial_x u)_0 \\ 0 \\ 0 \\ u_5 \end{bmatrix}$$

In matrix operator notation this is

$$B(4 : 2, 4, 0)\delta_x \mathbf{u} = \frac{1}{\Delta x} B(4 : -5, 4, 1)\mathbf{u} + (bc)$$

6. Setting $b = 0$ in problem 1, the Taylor table yields the following equation

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

This has the solution $[a, c, d] = [1, 2, -2]$, so the scheme can be expressed as

$$(\delta_x u)_{j-1} + (\delta_x u)_j - \frac{1}{\Delta x}[-2u_{j-1} + 2u_j] = O(\Delta x^2)$$

The Taylor series error for this scheme is

$$\begin{aligned} er_t &= \left(1 \cdot (-1)^2 \cdot \frac{1}{2} + 2 \cdot (-1)^3 \cdot \frac{1}{6}\right) \Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_j \\ &= \frac{\Delta x^2}{6} \left(\frac{\partial^3 u}{\partial x^3}\right)_j \end{aligned}$$